TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 150, July 1970

ELEMENTS WITH TRIVIAL CENTRALIZER IN WREATH PRODUCTS

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Abstract. Groups with self-centralizing elements have been investigated in recent papers by Kappe, Konvisser and Seksenbaev. In particular, if $G=A$ wr B is a wreath product some necessary and some sufficient conditions have been given for the existence of self-centralizing elements and for $G=\langle S_G\rangle$, where S_G is the set of self-centralizing elements. In this paper S_G and the set R_G of elements with trivial centralizer are determined both for restricted and unrestricted wreath products. Based on this the size of $\langle S_G\rangle$ and $\langle R_G\rangle$ is found in some cases, in particular if A and B are p-groups or if B is not periodic.

1. **Introduction.** An element x is said to have trivial centralizer in the group G if $\langle x, y\rangle$ is cyclic for each $y \in c_G x$. An element $x \in G$ is self-centralizing in G if $c_G x=\langle x\rangle$. Clearly self-centralizing elements have trivial centralizer but the converse is not true. The existence of a self-centralizing element x has a profound effect on the structure of the group. For example, if x is self-centralizing in G then the center of G is cyclic since $Z_1 G \subseteq c_G x=\langle x\rangle$, and there are other less obvious relations between some of the invariants of the group [1], [2], [3]. In many cases G is generated by the set S_G of all self-centralizing elements or the set R_G of all elements with trivial centralizer in G [2]. For the particular case of restricted wreath products Seksenbaev [5] has given some necessary and some sufficient conditions for $\langle S_G\rangle=G$ and $\langle R_G\rangle=G$, mainly for finite p-groups of odd order. In Theorem 1 of this paper we give a complete description of S_G and R_G for a wreath product $G=A$ wr B with nontrivial factors, and based on this we obtain the following results on the size of $\langle S_G\rangle$, $\langle R_G\rangle$ and a related group P_G for the restricted wreath product $G=A$ wr B of $A\neq 1$ and $B\neq 1$.

THEOREM 2. *Define $P_H=\langle xy \mid x, y \in S_H\rangle$ for any group H. If A and B are p-groups, B cyclic and $S_A\neq \varnothing$ then*

(a) $G/\langle S_G\rangle \cong A/A'P_A$.

(b) $|G:P_G|=2|A:A'P_A|$ *for* $p=2$.

Presented to the Society, January 25, 1969; received by the editors August 18, 1969.

AMS Subject Classifications. Primary 2052, 2054; Secondary 2040.

Key Words and Phrases. Wreath products, self-centralizing element, element with trivial centralizer, anticenter, p-group.

[1] Research of this author supported in part by a University of Cincinnati Faculty Fellowship.

(c) *Suppose* $B_i \neq 1$ *are cyclic p-groups and let* $A_1 = A$, $A_{i+1} = A_i \text{ wr } B_i$, $W = A_{k+1}$. *Then*

$$|W:P_W| = 2^k |A:A'P_A| \quad \textit{for } p = 2 \quad \textit{and} \quad W/\langle S_W \rangle \cong A/A'\langle S_A \rangle \quad \textit{for } p \neq 2.$$

THEOREM 3. *Let* B^* *be the subgroup generated by the elements of infinite order in* B *and let* $\bar{A}$ *be the base group of* $G = A \text{ wr } B$. *If* B *is not a torsion group then* $\langle S_G \rangle = \bar{A}B^*$.

2. **Definitions and notations.** Throughout this paper we will always assume that $A \neq 1$, $B \neq 1$ and $G = A \text{ wr } B$. Our notation for the standard wreath product essentially follows [4]. Let F be the group of all functions on B with values in A and define $f^b \in F$ for $b \in B$ and $f \in F$ by $f^b(x) = f(xb^{-1})$ for all $x \in B$. The unrestricted wreath product $A \text{ Wr } B$ is the semidirect product FB. The support of $f \in F$ is the set of all $x \in B$ with $f(x) \neq 1$. For a subgroup H of A we define

$$\bar{H} = \{f \in F \mid f(x) \in H \text{ for all } x \in B \text{ and } f \text{ has finite support}\}.$$

The restricted wreath product $A \text{ wr } B$ is the semidirect product $\bar{A}B$. The natural homomorphism of $A \text{ wr } B$ onto $A \text{ wr } B/\bar{A} \cong B$ is denoted by μ. For each $a \in A$ we define a function $\gamma_a \in \bar{A}$ by $\gamma_a(1) = a$ and $\gamma_a(x) = 1$ for all $x \neq 1$ in B. The mapping $\gamma\colon a \to \gamma_a$ is then an isomorphic embedding of A in $\bar{A}$ and $A \text{ wr } B$ is generated by A^γ and B.

For a given element $bf \in A \text{ wr } B$ with $b \in B$ and $f \in \bar{A}$ we define an element $h_x \in A$ for each $x \in B$ by

$$h_x = \prod_{i=-\infty}^{+\infty} f(xb^i) \quad \text{if } b \text{ has infinite order,}$$

$$h_x = f(xb)f(xb^2)\cdots f(xb^{|b|}) \quad \text{if } b \text{ has finite order.}$$

For any integer i we have $(bf)^i = b^i \cdot f^{b^{i-1}} \cdots f$. In particular, if b has finite order let $d = (bf)^{|b|} \in \bar{A}$. Then

$$d(x) = f(xb^{1-|b|})\cdots f(x) = h_x \quad \text{for all } x \in B.$$

3. **Some preliminary results.** In the following lemma some useful information about elements commuting with a fixed element $bf \in G = A \text{ wr } B$ is collected.

LEMMA 1. *Let* $b, c \in B$ *and* $g, f \in \bar{A}$. *Denote by* T *a left transversal of* $\langle b \rangle$ *in* B.

(i) $[cg, bf] = 1$ *if and only if* $g = f^{-c} g^b f$ *and* $[c, b] = 1$.

(ii) *If* b *has infinite order then* $[g, bf] = 1$ *implies* $g = 1$.

(iii) *Suppose* b *has infinite order and* $c \in c_B b$. *There exists an element* $g \in \bar{A}$ *such that* $[c^{-1}g, bf] = 1$ *if and only if* $h_{tc} = h_t$ *for all* $t \in T$.

(iv) *Assume* b *has finite order. For any* $a \in c_A h_t$ *and* $t \in T$ *define* $g_t \in \bar{A}$ *inductively by*

$$g_t(t) = a,$$

$$g_t(tb^i) = g_t(tb^{i-1})^{f(tb^i)} \quad \textit{for } 0 < i < |b|,$$

$$g_t(x) = 1 \quad \textit{for } x \notin t\langle b \rangle.$$

Then $[g_t, bf]=1$.

(v) *If* $h_t=1$ *for all* $t\in T$ *then* bf *and* b *are conjugate.*

(vi) *Assume* $g, g^*\in\bar{A}$ *commute with* bf, b *has finite order and* $g(t)=g^*(t)^\delta$ *for some integer* δ. *Then* $g(x)=g^*(x)^\delta$ *for all* $x\in t\langle b\rangle$.

REMARK 1. For later applications in §5 it should be noted that (i), (iv) and (v) provided b has finite order hold also in the unrestricted wreath product $X=A$ Wr B.

Proof. (i) $1=[cg, bf]\equiv[c, b] \bmod \bar{A}$ and $B\cap\bar{A}=1$ imply $[c, b]=1$. Hence

$$1 = [cg, bf] = g^{-1}c^{-1}f^{-1}b^{-1}cgbf = g^{-1}c^{-1}f^{-1}cb^{-1}gbf = g^{-1}f^{-c}g^bf.$$

Conversely if $[c, b]=1$ and $g=f^{-c}g^bf$ then $[cg, bf]=g^{-1}f^{-c}g^bf=g^{-1}g=1$.

(ii) If $[g, bf]=1$ then by (i), $g(x)=f^{-1}(x)g(xb^{-1})f(x)$ and by iteration $g(x)$ is conjugate to $g(xb^i)$ for all $x\in B$ and all integers i. Since g has finite support there is some x_t in each coset $t\langle b\rangle$ such that $g(x_t)=1$ hence $g=1$.

(iii) Suppose there is some $g\in A$ such that $[c^{-1}g, bf]=1$. From (i) we have $g(x)=f^{-1}(xc)g(xb^{-1})f(x)$ for all $x\in B$ hence by iteration for all $t\in T$ and all integers $j>0$

$$g(tb^j) = f^{-1}(tb^jc)\cdots f^{-1}(tb^{-j+1}c)g(tb^{-j})f(b^{-j+1})\cdots f(tb^j).$$

Since g and f have finite support and b has infinite order there exists an integer $N\geqq 0$ such that $1=f(tb^j)=f(tb^jc)=g(tb^j)$ for all $t\in T$ and all j with $|j|\geqq N$. Thus if $c\in c_Bb$ we have

$$\begin{aligned} 1 = g(tb^N) &= f^{-1}(tcb^N)\cdots f^{-1}(tcb^{-N+1})g(tb^{-N})f(tb^{-N+1})\cdots f(tb^N) \\ &= h_{tc}^{-1}h_t \qquad \text{for all } t\in T. \end{aligned}$$

Conversely assume $h_{tc}=h_t$ for all $t\in T$ and let $N\geqq 0$ be an integer such that $f(tb^j)=f(tb^jc)=1$ for all $t\in T$ and all j with $|j|\geqq N$. Define a function $g\colon B\to A$ inductively by

$$\begin{aligned} g(tb^{-j}) &= 1 \quad \text{for all } t\in T \text{ and all } j \geqq N, \\ g(tb^i) &= f^{-1}(tb^ic)g(tb^{i-1})f(tb^i) \quad \text{for } i > -N. \end{aligned}$$

By construction we have $g(x)=f^{-1}(xc)g(xb^{-1})f(x)$ for all $x\in B$ and $[c, b]=1$ by assumption, hence $[c^{-1}g, bf]=1$ by (i). To show $g\in\bar{A}$ observe that for $j\geqq N$ we have $g(tb^j)=g(tb^N)=h_{tc}^{-1}g(tb^{-N})h_t=h_{tc}^{-1}h_t=1$. Further

$$S = \{t \mid t\in T \text{ and } f(z) \neq 1 \text{ for some } z\in t\langle b\rangle\}$$

and

$$S^* = \{t \mid t\in T \text{ and } f(z) \neq 1 \text{ for some } z\in tc\langle b\rangle\}$$

are finite sets since $f(z)\neq 1$ for only finitely many $z\in B$ and Tc is also a left transversal of $\langle b\rangle$ in B if $[c, b]=1$. Thus $g(z)=1$ unless z belongs to the finite set of elements of the form

$$sb^j \qquad (s\in S;\ |j| \leqq N), \qquad s^*b^j \qquad (s^*\in S^*;\ |j| \leqq N).$$

This proves $g\in\bar{A}$ and hence (iii).

(iv) By (i) we have to prove $g_t(x)=f^{-1}(x)g_t(xb^{-1})f(x)$ for all $x\in B$. This is immediate from the definition for $x\notin t\langle b\rangle$ and $x=tb^i$ with $0<i<|b|$. For $x=t=tb^{|b|}$ we have inductively

$$f^{-1}(tb^{|b|})g_t(tb^{|b|-1})f(tb^{|b|}) = \cdots = g_t(t)^{f(tb)\cdots f(tb^{|b|})}$$
$$= g_t(t)^{h_t} = g_t(t),$$

since $g_t(t)=a\in c_A h_t$. Thus $g_t(x)=f^{-1}(x)g_t(xb^{-1})f(x)$ for all $x\in B$.

(v) If $b=1$ then $1=h_t=f(t)$ for all $t\in T$, hence $f=1$ and $1=bf=f$. Hence we may assume $b\neq 1$ and if b has finite order define $k\in\bar{A}$ inductively for all $t\in T$ by

$$k(t) = 1,$$
$$k(tb^i) = f^{-1}(tb^i)k(tb^{i-1}) \qquad \text{for } 0<i<|b|.$$

Then $(k^{-b}fk)(tb^i)=k^{-1}(tb^{i-1})f(tb^i)k(tb^i)=1$ for $0<i<|b|$. Further for $t=tb^{|b|}$ we have $(k^{-b}fk)(t)=k^{-1}(tb^{|b|-1})f(t)k(t)=k^{-1}(tb^{|b|-1})f(t)$ and $k(tb^{|b|-1})=f^{-1}(tb^{|b|-1})\cdots f^{-1}(tb)k(t)$. Hence $(k^{-b}fk)(t)=h_t=1$ and $k^{-b}fk=1$.

If b has infinite order there is some N such that $f(tb^i)=1$ for all $t\in T$ and all integers $|i|\geqq N$. Defining $k\in\bar{A}$ by $k(tb^{-i})=1$ for $i\geqq N$ and

$$k(tb^i) = f^{-1}(tb^i)k(tb^{i-1})$$

for $i>-N$ we will have $k(tb^i)=f^{-1}(tb^i)\cdots f^{-1}(tb^{-N})$, hence for all $i>N$

$$k(tb^i) = k(tb^{N+1}) = (f(tb^{-N})\cdots f(tb^N))^{-1} = h_t^{-1} = 1.$$

This proves $k\in\bar{A}$ and $k(tb^i)=f^{-1}(tb^i)k(tb^{i-1})$ for all i, and hence $(k^{-b}fk)(tb^i)=k^{-1}(tb^{i-1})f(tb^i)k(tb^i)=1$ for all integers i and all $t\in T$.

We have now $k^{-b}fk=1$ in both cases, and

$$b = (k^{-b}fk)b = b^{-1}k^{-1}bfkb = (bf)^{kb}$$

shows that b and bf are conjugate in G.

(vi) From Lemma 1(i) we get

$$g(x) = f^{-1}(x)g(xb^{-1})f(x) \quad\text{and}\quad g^*(x) = f^{-1}(x)g^*(xb^{-1})f(x)$$

and inductively that for each $x\in t\langle b\rangle$ there is some $a\in A$ such that $g(x)=g(t)^a$, $g^*(x)=g^*(t)^a$. Hence $g(x)=g(t)^a=g^*(t)^{\delta a}=(g^*(t)^a)^\delta=g^*(x)^\delta$.

LEMMA 2. (a) *A finite abelian group* $H=\langle a_1,\ldots,a_k,w\rangle$ *is cyclic if* $\langle a_i,w\rangle$ *is cyclic for* $i=1,\ldots,k$ *and* $(|a_i|,|a_j|)=1$ *for all* $i\neq j$.

(b) *The finite abelian group* $H=\langle u,v\rangle$ *is cyclic provided one of the following conditions is satisfied*:

(i) *H has a subgroup W such that* H/W *is cyclic and* $(|u|,|W|)=1$.

(ii) *There is some integer* $\alpha\neq 0$ *such that* $\langle u^\alpha,v\rangle$ *is cyclic and* $(|H/\langle u\rangle|,\alpha)=1$.

(c) *The abelian group* $H=\langle u,v\rangle$ *with* $u\neq 1$, $v\neq 1$ *is cyclic if and only if there exist integers* α,γ *with* $(\alpha,\gamma)=1$ *and* $u^\alpha=v^\gamma$.

Proof. (a) follows immediately from the main theorem for finite abelian groups. We note also the following consequence of the main theorem: if m and n are integers with $(m, n)=1$ then G is cyclic if and only if $G^m=\langle g^m \mid g \in G\rangle$ and $G^n=\langle g^n \mid g \in G\rangle$ are cyclic. Apply with $m=|W|$, $n=|u|$ for (i), with $m=|H/\langle u\rangle|$, $n=\alpha$ for (ii) and with $m=\alpha$, $n=\gamma$ for (c). For the converse in (c) let $H=\langle w\rangle$, $u=w^\sigma$, $v=w^\tau$ and ω the least common multiple of σ and τ. Then $\omega=\sigma\lambda=\tau\mu$ with $(\lambda, \mu)=1$ and $u^\lambda=w^\omega=v^\mu$.

LEMMA 3. *Let $g, f \in \bar{A}$, $1\neq b \in B$ and $g\neq 1$. If $\langle g, bf\rangle$ is cyclic then b has finite order and there are integers α, β with $(\alpha, |b|\beta)=1$ and $g^\alpha=d^\beta$, where $d=(bf)^{|b|}$.*

Proof. By Lemma 2(c) there are integers α, γ with $(\alpha, \gamma)=1$ and $g^\alpha=(bf)^\gamma$. Since $g \in \bar{A}$ and $\langle b\rangle \cap \bar{A}=1$ this implies that b has finite order and $|b|$ divides γ, hence $\gamma=|b|\beta$.

4. **Determination of S_G and R_G.** We now state and prove

THEOREM 1. *Let $f\in \bar{A}$ and $b \in B$ such that $bf\neq 1$ and let T be a left transversal of $\langle b\rangle$ in B.*

(a) *The element bf has trivial centralizer in $G=A$ wr B if and only if one of the following conditions is satisfied*:

(1)
- (1.1) *b has infinite order.*
- (1.2) *For each element $c \in c_Bb$ satisfying $h_{tc} = h_t$ for all $t \in T$ the subgroup $\langle c, b\rangle$ is cyclic.*

(2)
- (2.1) *b has finite order.*
- (2.2) *$h_t = 1$ for all $t \in T$.*
- (2.3) *$b \neq 1$ has trivial centralizer in B.*
- (2.4) *A is periodic and elements in A have order prime to $|b|$.*

(3)
- (3.1) *b has finite order.*
- (3.2) *$h_t \neq 1$ has trivial centralizer in A for all $t \in T$.*
- (3.3) *$c_Ah_t/\langle h_t\rangle$ is a p'-group for all primes p dividing $|b|$.*
- (3.4) *If $B \neq \langle b\rangle$, then c_Ah_t is periodic for all $t \in T$ and $(|h_s|, |y|)=1$ for all $s \neq t$ in T and all $y \in c_Ah_t$.*

(b) *The element bf is self-centralizing in G if and only if one of the following conditions is satisfied*:

(4)
- (4.1) *b has infinite order.*
- (4.2) *$c \in \langle b\rangle$ if and only if $c \in c_Bb$ and $h_{tc} = h_t$ for all $t \in T$.*

(5)
- (5.1) *b has finite order.*
- (5.2) *h_t is self-centralizing in A for all $t \in T$.*
- (5.3) *If $B \neq \langle b\rangle$, then h_t has finite order for all $t \in T$ and $(|h_s|, |h_t|)=1$ for all $s \neq t$ in T.*

REMARK 2. It should be noted that in cases (3) and (5) the group B is actually finite. Indeed, since f has finite support only finitely many $h_t = f(tb)\cdots f(tb^{|b|})$ are nontrivial, hence (3.2) or (5.2) imply that T and hence B is finite.

Proof. (1) Suppose bf has trivial centralizer, b has infinite order and $c \in c_B b$ satisfies $h_{tc} = h_t$ for all $t \in T$. From Lemma 1(iii) follows the existence of some $g \in \bar{A}$ such that $[c^{-1}g, bf] = 1$ and $bf \in R_G$ implies $\langle c^{-1}g, bf\rangle$ is cyclic. Let $H = \langle c, b\rangle$. Then $H^u = \langle c^{-1}g, bf\rangle^u$ is cyclic and $H^u = H\bar{A}/\bar{A} \cong H/H \cap \bar{A}$. But $H \cap \bar{A} \subseteq B \cap \bar{A} = 1$ so $H \cong H^u$ is cyclic.

Conversely suppose b and f satisfy conditions (1.1) and (1.2) and $w \in G$ commutes with bf, where $w = c^{-1}k$ for some $c \in B$ and $k \in \bar{A}$. Lemma 1(i) gives $c \in c_B b$ and $h_{tc} = h_t$ for all $t \in T$ from Lemma 1(iii), hence $\langle c, b\rangle$ is cyclic by (1.2). Let $K = \langle c^{-1}k, bf\rangle$. Then $K^u = \langle c, b\rangle^u$ is cyclic and $K^u = K\bar{A}/\bar{A} \cong K/K \cap \bar{A}$. But $[g, bf] = 1$ for all $g \in K \cap \bar{A}$ since K is abelian and so from Lemma 1(ii) we have $K \cap \bar{A} = 1$. Hence $K \cong K^u$ is cyclic.

(2) Suppose b has finite order and $h_t = 1$ for all $t \in T$. By assumption $bf \neq 1$. Then $b \neq 1$ since $b = 1$ implies $f(t) = h_t = 1$ for all $t \in T$, hence $f = 1 = bf$. To prove (2.3) observe that b is conjugate to $bf \in R_G$ by Lemma 1(v) and hence $b \in B \cap R_G \subseteq R_B$.

Finally for (2.4) let $a \in A$ and for each $t \in T$ define $k_t \in \bar{A}$ by

$$k_t(x) = a \quad \text{for } x \in t\langle b\rangle, \qquad k_t(x) = 1 \quad \text{for } x \notin t\langle b\rangle.$$

Then $[k_t, b] = 1$ by construction and $b \in R_G$ implies $\langle k_t, b\rangle$ cyclic. Hence k_t and a have finite order since $b \neq 1$ has finite order. If $r = (|a|, |b|)$ then $r = (|k_t|, |b|)$ and there are subgroups of order r in $\langle k_t, b\rangle$, $\langle k_t\rangle$ and $\langle b\rangle$. But there is only one subgroup H of order r in $\langle k_t, b\rangle$ since $\langle k_t, b\rangle$ is cyclic hence $H \subseteq \langle k_t\rangle \cap \langle b\rangle \subseteq \bar{A} \cap B = 1$. This proves $1 = r = (|a|, |b|)$ and thus (2.4).

Conversely assume conditions (2) are satisfied. Since b and bf are conjugate by (2.2) and Lemma 1(v) it is sufficient to show $b \in R_G$. Let $g \in \bar{A}$ and $c \in B$ be such that $[cg, b] = 1$. Then $[g, b] = 1 = [c, b]$ by Lemma 1(i) and $\langle c, b\rangle$ is a finite cyclic group by (2.3) and (2.1). Hence if $K = \langle cg, b\rangle$ then $K/K \cap \bar{A} \cong K\bar{A}/\bar{A} = K^u = \langle c, b\rangle^u$ is finite cyclic, $K \cap \bar{A}$ is finite abelian by (2.4) and $(|K \cap \bar{A}|, |b|) = 1$ by (2.4). Thus K is finite and Lemma 2(i) shows K cyclic.

(3) Assume b has finite order and $h_t \neq 1$ for some $t \in T$. Suppose $h_s = 1$ for some $s \in T$. Then $s \neq t$ and $h_t \in c_A h_s = A$. Define $g_s \in \bar{A}$ as in Lemma 1(iv) with $g_s(s) = h_t$. Then $\langle g_s, bf\rangle$ is abelian, hence cyclic and $g_s^\alpha = d^\beta$ with $(\alpha, \beta) = 1$ by Lemma 3. This gives $h_t^\alpha = 1$ for the argument s and $1 = h_t^\beta$ for the argument t, a contradiction since $(\alpha, \beta) = 1$ and $h_t \neq 1$.

To prove (3.2) let $a \in c_A h_t$ and define $g_t \in \bar{A}$ as in Lemma 1(iv) such that $g_t(t) = a$. Then $[bf, g_t] = 1$ and $\langle bf, g_t\rangle$ is cyclic since $bf \in R_G$. Hence also the subgroup $H = \langle d, g_t\rangle$ is cyclic, $H = \langle h\rangle$ with $h \in \bar{A}$, $d = h^i$ and $g_t = h^j$. So $h_t = d(t) = h(t)^i$, $a = g_t(t) = h(t)^j$ which proves that $\langle h_t, a\rangle$ is cyclic for all $a \in c_A h_t$.

To prove (3.3) we may assume $b \neq 1$ since the condition is vacuous for $b = 1$.

Defining $g_t \in \bar{A}$ as in Lemma 1(iv) for $1 \neq a \in c_A h_t$ with $g_t(t) = a$ we have $[g_t, bf] = 1$ and hence $\langle g_t, bf \rangle$ is cyclic. From Lemma 3 we obtain $g_t^\alpha = d^\beta$ with $(\alpha, |b|\beta) = 1$. For the argument t this gives $a^\alpha \in \langle h_t \rangle$, hence $c_A h_t / \langle h_t \rangle$ is periodic and $(\alpha, |b|) = 1$ proves (3.3).

Finally for (3.4) assume $B \neq \langle b \rangle$ and let $1 \neq y \in c_A h_t$. Define $1 \neq g_t \in \bar{A}$ as in Lemma 1(iv) with $g_t(t) = y$. Then $\langle g_t, bf \rangle$ is abelian, hence cyclic, and $g_t^\alpha = d^\beta$ with $(\alpha, \beta) = 1$ by Lemma 3. For the argument $s \neq t$ this gives $1 = h_s^\beta$, hence h_s has finite order for all $s \in T$. For the argument t we get $y^\alpha = h_t^\beta$. Since $(\alpha - \beta, \beta) = 1$ this implies $(|h_s|, |h_t|) = 1$ for $y = h_t \in c_A h_t$. But $|y|$ divides $\alpha |h_t|$ for all $y \in c_A h_t$, hence $(|h_s|, |y|)$ divides $(|h_s|, \alpha |h_t|) = (\beta, \alpha) = 1$.

Conversely assume that conditions (3) are satisfied, let $cg \in c_G(bf)$ and suppose $c \notin \langle b \rangle$. Then also $[cg, d] = 1$ where $d = (bf)^{|b|}$, and Lemma 1(i) gives $d = g^{-1} d^c g$ hence $d(x) = d(xc^{-1})^{g(x)}$ for all $x \in B$. Since $[d, bf] = 1$ we have also

$$d(x) = d(xb^{-1})^{f(x)} \quad \text{for all } x \in B,$$

so $d(xc^{-1})$ is conjugate to $d(z)$ for each $z \in x\langle b \rangle$. Let $tc^{-1} = sb^\beta$ with $s \in T$ and an integer β. Then $h_s = d(s)$ and $h_t = d(t)$ are conjugate, so (3.4) implies $s = t$, hence $c \in \langle b \rangle$, say $c = b^i$ for some integer i. Since $cg(bf)^{-i} \in \bar{A}$ and $\langle cg(bf)^{-i}, bf \rangle = \langle cg, bf \rangle$ we see that in order to prove that bf has trivial centralizer in G it suffices to show that $\langle g, bf \rangle$ is cyclic for each $g \in \bar{A}$ with $[g, bf] = 1$. Lemma 2(ii) with $u = bf$, $v = g$, $\alpha = |b|$ and $d = u^\alpha$ gives just that provided that we can show

(a) $\langle d, g \rangle$ is cyclic;

(b) $\langle bf, g \rangle / \langle bf \rangle$ is a p'-group for all primes p dividing $|b|$.

Since $\langle bf \rangle \cap \bar{A} = \langle (bf)^{|b|} \rangle = \langle d \rangle$ then

$$\langle bf \rangle \cap \langle g \rangle = \langle bf \rangle \cap \langle \bar{A} \rangle \cap \langle g \rangle = \langle d \rangle \cap \langle g \rangle.$$

Then

$$\langle bf, g \rangle / \langle bf \rangle \cong \langle g \rangle / \langle bf \rangle \cap \langle g \rangle = \langle g \rangle / \langle d \rangle \cap \langle g \rangle$$

shows that (b) can be replaced by

(b*) $\langle g \rangle / \langle d \rangle \cap \langle g \rangle$ is a p'-group for each prime p dividing $|b|$.

To show (a) assume first $B = \langle b \rangle$. Then $d(t) = h_t$ and (3.2) gives $\langle d(t), g(t) \rangle$ is cyclic, say $k_t = d(t)^\alpha g(t)^\beta$, $d(t) = k_t^i$ and $g(t) = k_t^j$ with integers α, β, i, j and $k_t \in A$. Let $k = d^\alpha g^\beta \in \bar{A}$ and observe that d, g and k commute with bf. From Lemma 1(vi) we get $d = k^i$ and $g = k^j$, so $\langle d, g \rangle = \langle k \rangle$ is cyclic.

For $B \neq \langle b \rangle$ the elements h_s, $g(s)$ have finite order for all $s \in T$ by (3.4). Since g commutes with bf we have from Lemma 1(i) that $g(s)$ and $g(x)$ are conjugate if $x \in s\langle b \rangle$. For each $t \in T$ define $d_t \in \bar{A}$ by $d_t(x) = d(x)$ for $x \in t\langle b \rangle$, $d(x) = 1$ for $x \notin t\langle b \rangle$. Then d_t commutes with d_s, g and bf, only finitely many d_t are nontrivial and d is the product of all d_t. Let Π be the set of all primes dividing $|h_t|$ and decompose $\langle g \rangle = \langle m \rangle \times \langle n \rangle$ with $m, n \in \bar{A}$ such that $\langle m \rangle$ is a Π-group and $\langle n \rangle$ is a Π'-group. Since $(|g(s)|, |h_t|) = 1$ by (3.4) for $s \neq t$ and $|g(x)| = |g(s)|$ for $x \in s\langle b \rangle$

it follows that $m(x)=1$ for $x \notin t\langle b\rangle$. By (3.2) the abelian group $\langle m(t), d_t(t)\rangle \subseteq \langle g(t), h_t\rangle$ is cyclic, say $\langle m(t), d_t(t)\rangle = \langle k_t\rangle$ with $k_t = m(t)^\alpha d_t(t)^\beta$. Let $k = m^\alpha d_t^\beta$, and observe that $m \in \langle g\rangle$, d_t and k commute with bf. Since $m(x)=d_t(x)=k(x)=1$ for $x \notin t\langle b\rangle$ we get then from Lemma 1(vi) that $\langle m, d_t\rangle = \langle k\rangle$ is a cyclic Π-group. Hence $\langle d_t, g\rangle = \langle d_t, m\rangle \times \langle n\rangle$ is cyclic. Finally $(|d_s|, |d_t|)=1$ for $s \neq t$ in T by(3.4), so Lemma 2(a) can be applied to prove (a).

To show (b*) assume first $B=\langle b\rangle$ and p divides $|b|$. Then by (3.3) there is some integer i such that $(p, i)=1$ and $g(t)^i \in \langle h_t\rangle$, say $g(t)^i = d(t)^j$. But Lemma 1(vi) gives $g(x)^i = d(x)^j$ for all $x \in B$ so $g^i \in \langle d\rangle \cap \langle g\rangle$.

If $B \neq \langle b\rangle$, then $c_A h_t$ is periodic for all $t \in T$ by (3.4). Suppose there is some prime p dividing $|b|$ and some $k \in \langle g\rangle$ such that $k^p \in \langle d\rangle$. In particular $k(t) \in \langle h_t\rangle$ for all $t \in T$ by (3.3), say $k(t) = h_t^{\alpha_t}$ with integers α_t. Now $(|h_s|, |h_t|)=1$ for $s \neq t$ in T by (3.4), and the Chinese remainder theorem gives the existence of an integer α such that

$$\alpha \equiv \alpha_t \bmod |h_t| \qquad \text{for all } t \in T.$$

Finally $d(t)=h_t$, so Lemma 1(vi) shows $k(x)=d(x)^\alpha$ for all $x \in B$. Hence there is no element of order p in $\langle g\rangle/\langle d\rangle \cap \langle g\rangle$, which proves (b*).

(4) Assume now that bf is a self-centralizing element in G and b has infinite order. If $c \in \langle b\rangle$ then $c \in c_B b$ and clearly $h_{xc}=h_x$ for all $x \in B$. Assume $c \in c_B b$ satisfies $h_{tc}=h_t$ for all $t \in T$. There exists $g \in \bar{A}$ by Lemma 1(iii) such that $[c^{-1}g, bf]=1$ hence $c^{-1}g \in \langle bf\rangle$ since bf is self-centralizing. If $c^{-1}g=(bf)^i$ then in particular $c^{-1}=b^i$ so b is self-centralizing.

Conversely, assume (4.1) and (4.2) are satisfied and there are $c \in B$ and $g \in \bar{A}$ such that $[c^{-1}g, bf]=1$. Then $[c, b]=1$ from Lemma 1(i) and Lemma 1(iii) together with (4.2) implies $c=b^j$ with some integer j. But $k=(bf)^{-j}(c^{-1}g) \in \bar{A}$ commutes with bf, hence $k=1$ by Lemma 1(ii) and so $c^{-1}g \in \langle bf\rangle$.

(5) Suppose b has finite order and let $a \in c_A h_t$. Define g_t as in Lemma 1(iv) with $g_t(t)=a$. Then $[g_t, bf]=1$ and by assumption $g_t=(bf)^i$ with some integer i. But $g_t \in \bar{A}$ implies that $|b|$ divides i so $i=|b|j$ and $(bf)^i=(bf)^{|b|j}=d^j$. Hence $a=g_t(t) = d(t)^j = h_t^j$ which proves (5.2).

Since a self-centralizing element has trivial centralizer we may apply the results of (2) and (3) to prove (5.3). We have $c_A h_t = \langle h_t\rangle$ by (5.2) hence, $h_t \neq 1$ for all $t \in T$ and condition (3.4) gives (5.3).

Conversely, assume (5) is satisfied, cg commutes with bf and $c \notin \langle b\rangle$. Then $d=(bf)^{|b|}$ commutes with cg and bf so Lemma 1(i) implies

$$d = f^{-1}d^b f \quad\text{and}\quad d = g^{-1}d^c g.$$

In particular, if $tc^{-1}=sb^\beta$ with $s \in T$ and an integer β, then $h_t=d(t)$ is conjugate to $d(s)=h_s$ and hence $s=t$ by (5.3). But then $c=b^{-\beta} \in \langle b\rangle$, a contradiction. Thus $c \in \langle b\rangle$, say $c=b^i$ and $k=(bf)^{-i}(cg) \in \bar{A}$ commutes with bf and d. Since $d(t)=h_t$ is self-centralizing by (5.2) there is an integer α_t for each $t \in T$ such that $k(t)=d(t)^{\alpha_t}$.

From (5.3) we have $(|d(s)|, |d(t)|)=1$ for $s\neq t$ in T, so by the Chinese remainder theorem there is an integer α with

$$\alpha \equiv \alpha_t \bmod |d(t)| \quad \text{for all } t\in T.$$

Then $k(t)=d(t)^{\alpha}$ for all $t\in T$ and Lemma 1(vi) implies $k(x)=d(x)^{\alpha}$ for all $x\in B$. Hence $k\in\langle d\rangle\subseteq\langle bf\rangle$, and bf is self-centralizing.

COROLLARY 1. *If B is a p-group and A is torsion-free or a p-group then $1\neq bf\in R_G$ with $b\in B$ and $f\in\bar{A}$ implies $B=\langle b\rangle$.*

Proof. Since B is periodic we have only to consider conditions (2) and (3). Further under our hypothesis it is impossible to satisfy condition (2.4). Hence from (3.2) we get $h_t\neq 1$ for all $t\in T$. Thus $(|h_s|, |h_t|)\neq 1$ if A is a p-group, and hence $B=\langle b\rangle$ by (3.4). If A is torsion-free we observe that $1\neq h_t\in c_Ah_t$, hence $1\neq c_Ah_t\subseteq A$ is not periodic and $B=\langle b\rangle$ follows again from (3.4). For later applications we note the following

COROLLARY 2. *If A and B are p-groups then $bf\in S_G$ if and only if $B=\langle b\rangle$ and $h_1\in S_A$.*

COROLLARY 3. *If A and B are p-groups then*

(i) $\{1\}\cup S_G=R_G$.

(ii) $S_G\neq\varnothing$ *if and only if B is cyclic and $S_A\neq\varnothing$.*

Proof. (i) Clearly $\{1\}\cup S_G\subseteq R_G$. So let $1\neq bf\in R_G$. From Corollary 1 we have $B=\langle b\rangle$, hence $b\neq 1$ and from (3.3) we see $c_Ah_t=\langle h_t\rangle$. Thus condition (5) is satisfied for bf and hence $bf\in S_G$.

(ii) Suppose $S_G\neq\varnothing$ and $bf\in S_G$. Then $B=\langle b\rangle$ and $S_A\neq\varnothing$ since $h_1\in S_A$ by Corollary 2. Conversely assume $B=\langle b\rangle$, $a\in S_A$ and let $f=\gamma_a$. Then

$$h_1 = f(b)\cdots f(b^{|b|}) = a\in S_A,$$

and $bf\in S_G$ by Corollary 2, hence $S_G\neq\varnothing$.

5. The size of S_G and P_G.

We introduce a new characteristic subgroup P_H which will be useful to compute S_G if G is an iterated wreath product.

DEFINITION. $P_H=\langle xy \mid x, y\in S_H\rangle$.

By definition $P_H\subseteq\langle S_H\rangle$, and if H is a p-group, $p\neq 2$ then $\langle x\rangle=\langle x^2\rangle$ shows $P_H=\langle S_H\rangle$. The generalized quaternion groups are examples for $P_H=\langle S_H\rangle$, while for the dihedral groups of 2-power order $P_H\neq\langle S_H\rangle$.

LEMMA 4. *Suppose A and $B=\langle b\rangle$ are p-groups and $S_A\neq\varnothing$. Then*

(i) $G'\subseteq P_G$.

(ii) $(A'P_A)^{\gamma}=A^{\gamma}\cap P_G=A^{\gamma}\cap\langle S_G\rangle$.

Proof. (i) Let $u, v, w\in A$, $uw=1$ and $1\neq c\in B=\langle b\rangle$. If $a\in S_A$ then $\gamma_u\gamma_{wa}^c b\in S_G$ by Corollary 2, hence

$$\gamma_u\gamma_w^c = \gamma_u\gamma_{wa}^c(\gamma_a^c)^{-1} = (\gamma_u\gamma_{wa}^c b)(\gamma_a^c b)^{-1}\in P_G.$$

But the elements $\gamma_u\gamma_w^c \subseteq P_G$ generate G' [4, Corollary 4.5, p. 350] hence $G' \subseteq P_G$.

(ii) If $a_1, a_2 \in S_A$ then $a_1^\gamma b \in S_G$ and $(a_2^{-1})^\gamma b \in S_G$ by Corollary 2 thus $(a_1a_2)^\gamma = a_1^\gamma a_2^\gamma = a_1^\gamma b((a_2^{-1})^\gamma b)^{-1} \in P_G$ hence $P_A^\gamma \subseteq P_G$. Since $(A')^\gamma \subseteq G'$ and $G' \subseteq P_G$ by Lemma 4(i) we have $(A'P_A)^\gamma \subseteq A^\gamma \cap P_G$.

Conversely let $h \in A^\gamma \cap P_G$. Then

$$h = (b_1f_1)\cdots(b_rf_r) \quad \text{where } r \text{ is even,}$$

$B = \langle b_1\rangle = \cdots = \langle b_r\rangle$ and $u_i = f_i(b_i)\cdots f_i(b_i^{|b|}) \in S_A$ by Corollary 2. Since

$$c'f_icf_{i+1} = c'cf_i^cf_{i+1} \quad \text{for every } c', c \in B$$

and $h \in A^\gamma \subseteq \bar{A}$ we may rewrite

$$h = f_1^*\cdots f_r^*,$$

with f_i^* conjugate to f_i under B. In particular $f_i^*(b)\cdots f_i^*(b^{|b|}) \equiv f_i(b)\cdots f_i(b^{|b|}) \equiv u_i$ mod A'. Now $h(b)\cdots h(b^{|b|}) \equiv \prod_{i=1}^r (f_i^*(b)\cdots f_i^*(b^{|b|})) \equiv u_1\cdots u_r$ mod A', and so $h(b)\cdots h(b^{|b|}) \in A'P_A$ since r is even. But $h \in A^\gamma$ implies $h(b^i) = 1$ for $0 < i < |b|$, so $h(1) = h(b^{|b|}) \in A'P_A$, or $h \in (A'P_A)^\gamma$. This proves $(A'P_A)^\gamma = A^\gamma \cap P_G$.

For $p \neq 2$ we have $P_G = \langle S_G\rangle$. To prove $A^\gamma \cap P_G = A^\gamma \cap \langle S_G\rangle$ we may hence assume $p = 2$. Suppose $h \in A^\gamma \cap \langle S_G\rangle$. By Corollary 2

$$h = (b_1f_1)\cdots(b_rf_r),$$

with $b_if_i \in S_G$, $B = \langle b\rangle = \langle b_1\rangle = \cdots = \langle b_r\rangle$ and $f_i \in \bar{A}$. Since $h \in A^\gamma \subseteq \bar{A}$ and $h \equiv b_1\cdots b_r$ mod $\bar{A}$, we have $b_1\cdots b_r \in \bar{A} \cap B = 1$. But $B = \langle b\rangle = \langle b_i\rangle$ and $p = 2$ imply that each b_i is an odd power of b, hence r is even because b has even order. Thus $h \in P_G$, which proves

$$A^\gamma \cap \langle S_G\rangle = A^\gamma \cap P_G,$$

since trivially $P_G \subseteq \langle S_G\rangle$.

We can now prove Theorem 2 announced in the introduction.

Proof. (a) We first observe that $G = A^\gamma\langle S_G\rangle$ since $\langle A^\gamma, b\rangle = G$ and $\gamma_a b \in S_G$ by Corollary 2 for $B = \langle b\rangle$ and $a \in S_A$. Hence $G/\langle S_G\rangle \cong A^\gamma/A^\gamma \cap \langle S_G\rangle$, and $A^\gamma \cap \langle S_G\rangle = (A'P_A)^\gamma$ by Lemma 4(ii) implies

$$G/\langle S_G\rangle \cong A/A'P_A.$$

(b) By Corollary 2 an element $x \in S_G$ has the form $x = b^*f$ with $f \in \bar{A}$ and $B = \langle b^*\rangle$ where b^* is an odd power of b. In particular $P_G \subseteq \bar{A}\langle b^2\rangle$. But $\gamma_a b \in S_G$ for $a \in S_A$ so $b^2 \in \bar{A}P_G$. Hence $\bar{A}P_G = \bar{A}\langle b^2\rangle$ and

$$|G:\bar{A}P_G| = |G:\bar{A}\langle b^2\rangle| = 2.$$

Further $A^\gamma P_G$ is normal in G since $G' \subseteq A^\gamma P_G$ by Lemma 4(i). But $\bar{A}$ is generated by conjugates of A^γ, hence $\bar{A}P_G = A^\gamma P_G$ and $\bar{A}P_G/P_G = A^\gamma P_G/P_G \cong A^\gamma/A^\gamma \cap P_G \cong A/A'P_A$ by Lemma 4(ii). This proves

$$|G:P_G| = |G:\bar{A}\,P_G|\;|\bar{A}P_G/\,P_G| = 2|A:A'P_A|.$$

(c) For $k=1$ this follows directly from (a) and (b). We proceed by induction on k. Let $V=A_{k+2}=W \operatorname{wr} B_{k+1}$. Then for $p=2$, $|V:P_V|=2|W:W'P_W|$ by (b) and $W'\subseteq P_W$ by Lemma 4(i). By induction $|W:P_W|=2^k|A:A'P_A|$ hence $|V:P_V|=2^{k+1}|A:A'P_A|$. For $p\neq 2$ we have $V/\langle S_V\rangle \cong W/W'P_W$ by (a), $P_W=\langle S_W\rangle$ and $W'\subseteq P_W$ by Lemma 4(i), hence $V/\langle S_V\rangle\cong W/\langle S_W\rangle$ and by induction

$$V/\langle S_V\rangle \cong W/\langle S_W\rangle \cong A/A'P_A.$$

Proof of Theorem 3. Let b be an element of infinite order in B, $1\neq a\in A$ and choose T so that $1\in T$. We show first that if $f=\gamma_a$, then the element bf satisfies condition (4.2) of Theorem 1. Observe that

$$h_x = 1 \quad \text{for } x\notin\langle b\rangle,$$
$$h_x = a \quad \text{for } x\in\langle b\rangle.$$

Hence $h_{tc}=h_t$ implies for $t=1$ that $h_{tc}\neq 1$ and so $c\in\langle b\rangle$. This proves by Theorem 1(4) that $b\gamma_a\in S_G$ and a similar argument shows $b^2\gamma_a\in S_G$ hence both b and γ_a are contained in $\langle S_G\rangle$. But this implies $B^*\subseteq\langle S_G\rangle$ and $A^\gamma\subseteq\langle S_G\rangle$. Since $\bar{A}$ is generated by the conjugates of A^γ and $\langle S_G\rangle$ is normal in G, we have $\bar{A}B^*\subseteq\langle S_G\rangle$.

On the other hand, if $bf\in S_G$ then $b\in B^*$ for otherwise b has finite order and Remark 2 implies B is finite which contradicts the assumption that B is not a torsion group. Thus $S_G\subseteq\bar{A}B^*$, hence $\langle S_G\rangle=\bar{A}B^*$.

6. **Unrestricted wreath products.** The characterization of self-centralizing elements and elements with trivial centralizer is much easier for the unrestricted wreath product as the following theorem shows. In particular if A and B are p-groups it suffices to consider the restricted wreath product.

THEOREM 4. *Suppose B has infinite order and $X=A \operatorname{Wr} B$ is the unrestricted wreath product of A and B.*

(a) *Let $b\in B$ and $f\in F$ such that $bf\neq 1$, and T a left transversal of $\langle b\rangle$ in B. Then bf has trivial centralizer in X if and only if the following conditions are satisfied*:

(i) *b has finite order.*

(ii) *$h_t=1$ for all $t\in T$.*

(iii) *$b\neq 1$ has trivial centralizer in B.*

(iv) *A is periodic and elements in A have order prime to $|b|$.*

(b) *X has no self-centralizing elements.*

Proof. (a) Suppose $bf\in R_X$. If b has infinite order, let $1\neq a\in A$ and define $1\neq g\in F$ by

$$g(1) = a, \qquad g(x) = g(xb^{-1})^{f(x)} \quad \text{for } x\in\langle b\rangle$$

and $g(x)=1$ for $x\notin\langle b\rangle$. Then by construction $g^{bf}=g$, so that $\langle g, bf\rangle$ must be cyclic. This contradicts Lemma 3 and this proves (i).

Let $1\neq a\in c_A h_t$ and define g_t as in Lemma 1(iv) with $g_t(t)=a$. Then $\langle g_t, bf\rangle$ is cyclic and by Lemma 3 $g_t^\alpha = d^\beta$ with integers α, β such that $(\alpha, |b|\beta) = 1$. In particular

(*) $$a^\alpha = h_t^\beta, \qquad 1 = h_s^\beta \quad \text{for all } s \neq t \text{ in } T.$$

Hence h_s has finite order, $c_A h_t$ is periodic and there is some integer $m > 0$ such that $h_s^m = 1$ for all $s \in T$. With $a = h_t$ we see from (*) also that $(|h_s|, |h_t|) = 1$ since $(\alpha - \beta, \beta) = 1$. Since $h_s^m = 1$ for all $s \in T$ this implies that there exists some $t \in T$ with $h_t = 1$. Suppose $h_s \neq 1$ for some $s \in T$. Then for $a = h_s \in A = c_A h_t$ (*) implies $h_s^\alpha = 1$ and $1 = h_s^\beta$ a contradiction since $(\alpha, \beta) = 1$. This proves (ii).

To prove (iii) observe that bf and b are conjugate by (ii) and Lemma 1(v). In particular $b \neq 1$ and $b \in R_X \cap B \subseteq R_B$.

Finally from (ii) and (*) we get $a^\alpha = 1$ for $a \in c_A h_t = A$ with $(\alpha, |b|\beta) = 1$ which proves (iv).

The sufficiency of conditions (i) to (iv) follows as in the proof of Theorem 1(2).

(b) Since $S_X \subseteq R_X$ we get from condition (ii) of (a) that $h_t = 1$ for all $t \in T$. Then Lemma 1(v) implies that bf and b are conjugate, hence $b \in S_X$. Then for some $1 \neq a \in A$ define $k \in \bar{A} \subseteq F$ by $k(x) = a$ for $x \in \langle b \rangle$ and $k(x) = 1$ for $x \notin \langle b \rangle$, and observe $[k, b] = 1$, but $\langle b \rangle \cap F \subseteq B \cap F = 1$.

Corollary 4. *If A and B are p-groups then $R_X = \{1\} \cup S_X$.*

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